In Honor of The Nobel Laureates Robert C. Merton And Myron S. Scholes: A Partial Differential Equation that Changed the World

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The origin of modern option pricing theory is Bachelier's (1900) dissertation on the theory of speculation. Bachelier derived an option pricing formula that today can be recognized as a variation to the Black-Merton-Scholes formula.

He also developed some necessary mathematics related to diffusion processes and Brownian motions. This dissertation, motivated both Samuelson's (1965) paper on option pricing and Ito's lemma which provides an essential step in the derivation of the Black-Merton-Scholes formula. Papers published by Sprenkle (1961) and Merton and Samuelson (1969), determined an option's price using the maximizing conditions obtained from an investor's optimal portfolio position. As such, their valuation formulas depended on the expected return on the stock or equivalently, the stock's risk premium. This dependency made these formulas difficult to estimate and to use as the risk premia shifted according to changing tastes and economic fundamentals. The second difficulty, was that the valuation formulas offered no sense of how to hedge an option using a portfolio of the underlying stock and riskless borrowing or lending.

Fischer Black tried to apply the capital asset pricing model to value the option in a continuous time setting. He obtained an implicit solution for the option's value characterized as a partial differential equation, but could not find its solution. He then teamed up with Myron Scholes, to solve the equation.

Later, Merton showed Black and Scholes how to derive their partial differential equation differently. Merton's derivation used only an argument based on the continuous-time construction of a perfectly hedged portfolio involving the stock and the call option, and the notion that no arbitrage opportunities exist.

A brief explanation of how options work is in order here. A call option on the stock increases in value when the stock price rises. This is because as the stock price rises, the stock price is more likely to lie above the strike price at maturity. Hence, the call option is more valuable today. A short position in the stock can be used to hedge against changes in the value of a call option to its holder. If the stock price rises over a short period, the option's value will increase but the short position in the stock will decrease in value. These changes in value partially offset each other, hence, a partial hedge.

Conversely, if the stock price falls over a short time period, the value of the option decreases in value, but the short position increases in value. Again, this yields a partial hedge. The partially hedged position in the long call option and short stock can be modified to make it an exact hedge over a short time period. Indeed, it is possible to determine the exact number of shares of stock (less than one) to short for each long call option, so that for any change in the stock price, the change in the value of the call option is exactly offset by the change in value of the short position in the stock. This gives a perfectly hedged portfolio. That way, we can also uniquely identify the option's arbitrage free price. Why? The portfolio requires a known initial investment and it is riskless over a given time period. Hence, to avoid arbitrage, it must earn the riskless rate. Otherwise, profits could be made by buying the hedged portfolio and selling the riskless asset, or vice versa. This restriction determines the change in the value of the call option as a function of the value of the underlying stock price and the riskless rate. More precisely, it leads to a partial differential equation satisfied by the call's value, whose solution is the Black-Merton-Scholes formula.

The Black-Merton-Scholes formula does not explicitly depend on the expected return on the stock or, equivalently, the stock's risk premium, unlike the option price formulations of the early and mid-1960s. Embedded within the derivation of the Black-Merton-Scholes formula are two key assumptions. The first is that the risk-free interest rate is constant. The second is that the stock price's distribution has a 2 constant volatility. Both assumptions are reasonable for short-term (a year or less) options on equities or equity indices whose returns are uncorrelated with changes in interest rates.

Two concepts underlie the Black-Merton-Scholes option pricing argument: "no-arbitrage" and "complete markets." No-arbitrage has already been covered. A complete market is one where the synthetic construction of any security or derivative is possible.

The "no-arbitrage" and "complete market" concepts were first formalized in papers by Harrison and Kreps (1979) and Harrison and Pliska (1981) who showed that, these concepts could be characterized using the notion of a martingale probability distribution. A martingale is a stochastic process whose expected future value equals its current value. Essentially, the concept that no arbitrage possibilities exist is equivalent to the notion that the stock's expected future value is the same as its current price (appropriately discounted), which essentially means that the stock is properly priced. Any changes in the stock's price through time are caused by unanticipated and random events.

The increased volatility of interest rates in later years, created a new demand for interest rate derivatives, for both hedging and speculation. The assumption of constant interest rates needed to be relaxed. This extension was provided by Ho and Lee (1986), Black, Derman and Toy (1990), and Heath, Jarrow and Morton (1992). They showed that there is an evolution of multiple term structures of futures prices or interest rates upon which the derivatives are written. If the no-arbitrage condition holds, analogous to the Black-Merton-Scholes formula, then option prices only depend on the term structure's initial values and volatilities. These later derivative applications were those implemented in the 1990s.

Contingent claims analysis is the application of option pricing theory to the valuation of corporate liabilities like debt, equity and convertible bonds. The firm's equity can be considered as a European call option on the firm's assets, with a strike price equal to the face value of the debt and a maturity date equal to the maturity of the debt. This notion is simple, but powerful. It implies that the debtholders are actually the ultimate owners of the firm's assets, having written a call option on them to the equity holders. Of course, this contrasts with the traditional finance perspective that the sole owners of the firm are the equity holders. Contingent claims analysis argues that both the debt and equity holders "own" the firm. The debtholders get the first stream of payments from the firm's assets and the equity holders get the residual.

Since the equity is a European call option on the value of the assets, we know from the Black-Merton-Scholes formula that the value of equity increases with the volatility of the firm's asset returns. Essentially, this occurs because greater volatility raises the chance of an extremely good outcome for equity holders. At the same time, since the losses of equity holders are bounded at zero, they do not need to worry about whether losses below a certain level are slight or considerable. Consequently, once debt is in place, equity holders have an incentive (acting through management) to increase the risk of the firm's assets, thereby increasing the value of their equity and decreasing the value of the debt. This is nothing but the agency cost of debt.