Note
Managing Market Risk
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"The best traders are not right more than they are wrong. They are quick adjusters. They are better at getting right when they are wrong."

- Lloyd Blankfein,
CEO, Goldman Sachs[1]

Introduction

Market risk can be understood in simple terms as the possibility of losses arising due to fluctuations in currency rates, interest rates, equity markets and commodity prices. By its very nature, market risk is amenable to quantification. Indeed, this is the world of statistics and probability distributions. Data on currencies, interest rates, stock markets and commodity markets are readily available either in the public domain or with specialised vendors. Using this data and making suitable assumptions about the underlying probability distributions, we can answer questions such as "What is our maximum loss likely to be say 90% of the time?" But as in many other uncertain situations, we must be careful while handling data. The current pattern of a market variable may change in the future. Unexpected, worst case scenarios may result. This is exactly what happened as we moved from a benign period into the sub prime crisis. Moreover, the choice of probability distribution is critical. A wrong distribution will obviously lead to erroneous results. We will try to understand the building blocks of market risk management in this chapter – Volatility, Stock price modeling techniques, Ito's Lemma, Risk Neutral valuation and Black Scholes Option Pricing Model.

Volatility: The basics

Volatility is probably the most important concept in market risk management. In simple terms, volatility is the uncertainty about the return provided by the underlying asset. The higher the volatility, the higher the market risk. At this stage, we can define volatility loosely as the standard deviation of the return provided by the underlying asset. To measure volatility, we observe the price of the underlying asset at definite intervals of time, say daily.

If S_i is the price of the asset at the end of the day and S_{i-1} at the beginning of the day, we can define the daily return as $\frac{S_i - S_{i-1}}{S_i}$. If we use continuous compounding, the daily return becomes $ln(S_i/S_{i-1})$. A brief explanation is in order here.

Let r be the daily return. If we use a simple measure of return, we can write:

$$S_i = S_{i-1} (1 + r)$$

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[1] William D. Cohan, "The rage over Goldman Sachs," Time.com, August 31, 2009.

But if we assume the day is divided into *m* intervals and the returns are compounded *m* times during the day, we can write:

$$S_i = S_{i-1} \left(1+\frac{r}{m}\right)^m$$

or

$$S_i = S_{i-1}\left[\left(1+\frac{r}{m}\right)^{m/r}\right]^r$$

As *m* becomes very large, we move towards continuous compounding,

But as *m* becomes very large, we also know from basic calculus that:

$$\left(1+\frac{r}{m}\right)^{m/r} = e$$

So we can write, in the case of continuous compounding,

$$S_i = S_{i-1}e^r$$

or

$$\frac{S_i}{S_{i-1}} = e^r$$

Taking natural logarithms on both sides we get

$$ln \frac{S_i}{S_{i-1}} = ln e^r$$

or

$$ln \frac{S_i}{S_{i-1}} = r \qquad \text{(because } (ln e^r = r\, ln e = r)\text{)}$$

Since we have worked out *r* for a specific day, we can write $ln \frac{S_i}{S_{i-1}} = r_i$.

Thus the daily continuously compounded rate of return is nothing but the logarithm of the ratio of the closing price for the day to the opening price.

Illustration

The stock price at the beginning of a trading day is 40 and at the end of the day it is 41.

Then the continuously compounded daily rate of return $= ln \frac{41}{40} = .0247$

Whereas the simple compounded daily rate of return $= \frac{41-40}{40} = .0250$

If we measure the daily returns over *n* days, we will get *n* data points. Let the mean be $\bar{r}$. The standard deviation of these returns is nothing but the volatility. This is given by the well known formula in statistics:

$$\sigma^2_n = \frac{1}{n-1} \left[(r_1 - \bar{r})^2 + (r_2 - \bar{r})^2 + \ldots + (r_n - \bar{r})^2\right]$$

$$= \frac{1}{n-1}\sum_{i=1}^{n}(r_i - \bar{r})^2$$

where σ_n is the volatility.

We use n-1, not n, because while calculating the standard deviation of a sample, there are only n-1, degrees of freedom. We have "consumed" one degree of freedom while computing the sample mean. But in many cases "*n*" is a good approximation for "n-1."

For a good estimate of volatility[2], n should be reasonably large, at least 30 and preferably 100. If we use daily trading returns we need to get the data for about 5 months. If we use weekly returns we need to get the data for about 2 years. And if we use monthly returns, we would need the data for about 8 years. Over long periods of time, volatility is unlikely to remain constant. During 5 months, it is reasonable to assume that volatility will not change. That is why, volatility is usually calculated using daily returns and then scaled up suitably. We will examine shortly how this scaling is done.

Illustration

Consider a sum of Rs. 100. The interest rate is 10% per annum. What will be the amount after one year, if the compounding is done annually, semi annually, quarterly, daily and continuously?

Principal = Rs. 100. Interest rate = 10%

Compounding	*Amount*
Annual compounding	= 100 (1.1) = 110
Semi Annual	$= 100\ (1.05)^2$ = 110.25
Quarterly	$= 100\ (1.025)^4$ = 110.38
Daily	$= 100\ (1+.1/365)^{365}$ = 110.52
Continuous	$= 100\ (1+.10/m)^m$ $= 100(1+.10/m)^{m/.10}]^{.10}$ $= 100e^{.10}$ = 110.52

Illustration

How high must the continuously compounded interest rate r be for the same amount to accumulate as with annual compounding at interest rate R?

Let the capital invested be V_0 and the time period of investment be T

$$V_0 \cdot (1 + R)^T = V_0 \cdot e^{rT}$$

[2] Jayanth Rama Varma, "Derivatives and Risk Management," Tata McGraw Hill, 2008.

As a result:
$(1 + R)^T = e^{rT}$
or $T\ ln(1 + R) = rT\ lne = rT$
$r = ln(1 + R)$

Illustration
Consider a stock whose current price is 20 and expected return is 20% per annum. What is the expected stock price, E (S_T) in 1 year ?

$E(S_T) = S_0e^{\mu T} = (20)e^{(.20)(1)} = 24.43$

Illustration
The stock prices at the end of 5 consecutive days of trading are 40, 41, 42, 41 and 40. How do we estimate the daily volatility?

First we calculate the daily continuously compounded rate of return. These will be $ln\frac{41}{40}$, $ln\frac{42}{41}$, $ln\frac{41}{42}$ and $ln\frac{40}{41}$. Next we calculate the mean and standard deviation as indicated in the table below:

Day	r	$r-\bar{r}$	$(r-\bar{r})^2$
1	.0247	.0247	**.00061**
2	.0241	.0241	**.00058**
3	- .0241	- .0241	**.00058**
4	- .0247	- .0247	**.00061**
	$\bar{r}=0$		**.00238**

So the standard deviation of the daily returns is $\sqrt{.00238/3} = .0282$.
(Note that we divided by 3, not 4.)
This is nothing but the volatility.

Scaling Volatility
If the daily volatility is known, how do we calculate the weekly volatility? We can understand intuitively that the longer the time period, the wider the range of movement in the underlying variable. In a day, the value of the variable can move only so much but in a week, it can move more, in a month, it can move even more and so on.

Will this increase in volatility be linear? To answer this question, we can view a one week period as a combination of seven days. So the weekly distribution of returns is the sum of seven daily distributions. If we assume the pattern of movements of the underlying does not change, we can assume all the distributions are identical. Let us also assume that these distributions are independent.

To get the weekly distribution, we have to aggregate the daily distributions. When we do that, the means and variances can be added. So the weekly return will be 7 times the daily return. The weekly variance will be 7 times the daily variance. But volatility refers to the standard deviation, not variance. And standard deviation is the square root of variance. So we can say that the weekly volatility is √7 times the daily volatility. Similarly if the weekly volatility is σ, we could state that the daily volatility is $\frac{\sigma}{\sqrt{7}}$. If the annual volatility is σ, we can estimate the daily volatility as $\frac{\sigma}{\sqrt{365}}$ and the weekly volatility as $\frac{\sigma}{\sqrt{52}}$. Similarly, we could state that the standard deviation of the stock price in 9 weeks is approximately 3 times the standard deviation in 1 week.

Let us derive the scaling rule more formally in the simple case of two trading days. Let the asset prices at different points of time, on the two days be x_1, x_2… x_n and y_1, y_2 …y_n respectively. Let the volatility of daily returns for the two days be σ_x and σ_y and the average of the daily return be $\bar{x}$ and $\bar{y}$ respectively for the two days considered separately.

Two day variance = $\frac{1}{n}\sum(x_i+y_i)^2-(\overline{x_i+y_i})^2$

Now σ_x^2 = $\frac{1}{n}\sum x_i^{\,2}-\bar{x}^2$

or $\frac{1}{n}\sum x_i^{\,2}=\bar{x}^2+\sigma_x^{\,2}$ = $\sigma_x^2+\bar{x}^2$

and σ_y^2 = $\frac{1}{n}\sum y_i^{\,2}-\bar{y}^2$

or $\frac{1}{n}\sum y_i^{\,2}=\bar{y}^2+\sigma_y^{\,2}$ = $\sigma_y^2+\bar{y}^2$

But $\overline{x_i+y_i}$ = $\bar{x}+\bar{y}$

Also σ_{xy}^2 = $\frac{1}{n}\sum xy-\bar{x}\,\bar{y}$

or $\frac{1}{n}\sum xy=\bar{x}\,\bar{y}+\sigma_{xy}^{\,2}$ = $\sigma_{xy}^2+\bar{x}\,\bar{y}$

Where $\overline{x_i+y_i}$ = mean of the combined distribution = $\bar{x}+\bar{y}$

σ_{xy}^2 = covariance of x, y

Expanding the variance term for the combined distribution, we get:

$$\frac{1}{n}\sum x_i^2 + \frac{1}{n}\sum y_i^2 + \frac{2}{n}\sum x_i y_i - \left(\overline{x_i + y_i}\right)^2$$

Substituting the expressions above we can rewrite this as:

$$\sigma_x^2 + \bar{x}^2 + \sigma_y^2 + \bar{y}^2 + (\sigma_{xy}^2 + \bar{x}\,\bar{y}) - (\bar{x} + \bar{y})^2$$

But $\sigma_{xy} = 0$ if the distributions are independently distributed.
So we can write:

Two day variance $= \sigma_x^2 + \bar{x}^2 + \sigma_y^2 + \bar{y}^2 + 2\,\bar{x}\,\bar{y} - \bar{x}^2 - 2\,\bar{x}\,\bar{y} - \bar{y}^2$
$= \sigma_x^2 + \sigma_y^2$

If the two distributions are assumed to be identically distributed, we can write:
$\sigma_x^2 = \sigma_y^2 = \sigma^2$

We get the variance of the combined distribution for 2 days as $\sigma^2 + \sigma^2 = 2\sigma^2$ and the standard deviation as $\sqrt{2}\sigma$. Instead of 2 days, if it had been *t* days, the standard deviation would have been $\sqrt{t}\sigma$.

Research indicates that volatility is much higher during trading days. So while estimating volatility, we ignore the holidays. So we can write:

Annual volatility = Daily Volatilty $\sqrt{No.of\ trading\ days\ per\ annum}$

It must be noted that the scaling approach discussed so far will not work if the movements in asset prices on different days are correlated. This is called *auto correlation*, i.e., correlation between variables of the same time series data.

A brief note about "scaling" the mean is also in order here. Using the earlier nomenclature, change in daily price of the underlying is $\Delta S_i = S_{i+1} - S_i$. The daily return is nothing but the change in price during the day divided by the initial price. To move from daily to annual return we have to divide by Δt. ie. $\mu = \frac{\Delta S_i}{S_i}\frac{1}{\Delta t}$ or $\mu\,\Delta t = \frac{\Delta S_i}{S_i}$ where μ is the annual return. Δt the time interval is 1/260, if we assume 260 trading days in year.

Illustration

Let us illustrate what we have covered so far.

a) Suppose the daily volatility is 3%. What is the 9 day volatility?

9 day volatility= 3% x √9 = 9%

b) Say the 25 day volatility is 10%. What is the daily volatility?

Daily volatility = $\frac{10\%}{\sqrt{25}}$ = 2%

More about volatility

We saw earlier that the daily volatility can be calculated as

$$\sigma_n^{\,2} = \frac{1}{n-1}[(r_1 - \bar{r})^2 + (r_2 - \bar{r})^2 + ...]$$

$$= \frac{1}{n-1}[(r_1^{\,2} - 2r_1\bar{r} + \bar{r}^2 + r_2^{\,2} - 2r_2\bar{r} + \bar{r}^2 + ...]$$

$$= \frac{1}{n-1}[(r_1^{\,2} + r_2^{\,2} + r_3^{\,2} + ...) - (2\bar{r})(r_1 + r_2 + r_3 + ...) + (\bar{r}^2 + \bar{r}^2 + ...]$$

$$= \frac{1}{n-1}[(r_1^{\,2} + r_2^{\,2} + r_3^{\,2} + ...) - 2n\bar{r}^2 + n\bar{r}^2]$$

or $$\sigma_n^{\,2} = \frac{1}{n-1}[(r_1^{\,2} + r_2^{\,2} + ...r_n^2) - n\bar{r}^2]$$

where *r* is the average return.

During a day, the upward and downward movements may cancel out. In other words, the average of the daily change in price may be small compared to the standard deviation. We will appreciate this more if we keep in mind that the standard deviation is calculated by squaring the deviations from the mean and taking their average and then finding the square root. Because of the squaring, the negative numbers become positive. So upward and downward deviations do not cancel out. As a result, the standard deviation may be significantly large compared to the average of the daily change.

Assuming $\bar{r}$ = 0($\bar{r}$ is the average return)

$$\sigma_n^{\,2} = \frac{1}{n-1}[(r_1^{\,2} + r_2^{\,2} ..... + r_n^{\,2}]$$

If we assume that n-1 ≈ n, we could also write:

$$\sigma_n^{\,2} \approx \frac{1}{n}\sum_{i=1}^{n} r_i^{\,2}$$

This is a formula that is easy to remember! It gives us a simple way to estimate volatility. But this is indeed too simple. We are giving equal weights to all the daily returns. We can improve the accuracy of our volatility estimates by using weighting schemes. Thus we could write:

$$\sigma_n^{\,2} = \sum_{i=1}^{m} \alpha_i r_{n-1}^{\;\;2}$$

Where α_i represents the weight and $\Sigma\alpha_i = 1$. Let us now look at some of these weighting schemes.

Exponentially Weighted Moving Average Model

One commonly used weighting scheme is the exponentially weighted moving average model (EWMA) which states:

$$\sigma_n^2 = \lambda\,\sigma_{n-1}^2 + (1-\lambda)\,r_{n-1}^2$$

where σ_n is the estimate of volatility we are trying to arrive at while σ_{n-1} and r_{n-1} are the latest estimates available of the daily volatility and daily return respectively. λ is a figure which we arrive at on the basis of our understanding of the past behaviour of the daily returns of the asset under consideration.

But $\sigma_{n-1}^2 = \lambda\,\sigma_{n-2}^2 + (1-\lambda)\,r_{n-2}^2$

So $\sigma_n^2 = \lambda\,[\lambda\sigma_{n-2}^2 + (1-\lambda)\,r_{n-2}^2] + (1-\lambda)\,r_{n-1}^2$

$$= \lambda^2\sigma_{n-2}^2 + \lambda(1-\lambda)r_{n-2}^2 + (1-\lambda)r_{n-1}^2$$

$$= \lambda^2\sigma_{n-2}^2 + (1-\lambda)\,[r_{n-1}^2 + \lambda r_{n-2}^2]$$

$$= \lambda^2[\lambda\sigma_{n-3}^2 + (1-\lambda)\,r_{n-3}^2] + (1-\lambda)\,[r_{n-1}^2 + \lambda r_{n-2}^2)]$$

$$= \lambda^3\sigma_{n-3}^3 + (1-\lambda)\,[r_{n-1}^2+\lambda r_{n-2}^2+\lambda^2 r_{n-3}^2]$$

To use the exponentially weighted model, we need to have only the current estimate of the variance rate and the most recent observation on the value of the market variable. As we get new observations, we can update the data. The value of λ determines how responsive the estimate of the volatility is to the most recent daily percentage change. A low value of λ means a lot of weight is being given to the previous day's change in price. If λ is high, less weight is given. This is because r_{n-1}^2 is multiplied by $1-\lambda$. A high value of λ also means the most recent value of standard deviation largely represents the volatility.

Illustration

The most recent estimate of the daily volatility is 10% and that of the daily return is 4%. What is the updated estimate of volatility if λ is .7?

Updated estimate of variance = $(.7)\,(.1)^2 + (1 - .7)\,(.04)^2$ = .00748

Updated estimate of volatility = $\sqrt{.00748}$ = .0865

The EWMA model comes in very handy when there is volatility clustering. Days of high volatility may tend to occur in clusters. In such situations, the EWMA becomes a good predictor of volatility because of the high weightage attached to the most recent volatility estimate.

The GARCH[3] Model

The GARCH model is a further improvement on the exponentially weighted moving average model. The advantage of the GARCH model is that it recognizes that variance rates are *mean reverting*. Mean reversion means that over time the value of a variable comes back to the average[4]. That means that if the value is too high, it will tend to come down. If it is too low, it will tend to go up. The GARCH model assumes that over time, the variance comes back to a long run average level of V_L.

This model can be represented as:

$$\sigma_n^2 = \gamma V_L + \alpha r_{n-1}^2 + \beta \sigma_{n-1}^2$$

V_L is the long run average variance rate.
$\alpha + \beta + \gamma = 1$.

When the current volatility is above the long term volatility, the Garch model estimates a downward sloping volatility term structure. (Term structure refers to the variation of volatility over time.) When the current volatility is below the long term volatility, it estimates an upward sloping volatility term structure. This is because when volatilities are currently low, there is an expectation that they will go up in future. When current volatility is high, there is an expectation that volatility will decrease in the future.

Illustration

The latest estimate of the volatility of daily returns is .08, that of the daily returns is .01 and the long term variance rate is .01.

If α = .3, β = .3 and γ = .4 find the updated estimate of the volatility.

Updated estimate of variance = $(.4)(.01) + (.3)(.01)^2 + (.3)(.08)^2$ = .00595

Updated estimate of volatility = $\sqrt{.00595}$ = .0771

[3] GARCH stands for Generalised Autoregressive Conditional Heteroskedasticity.
[4] Recall Francis Galton's pioneering work in Chapter 1.

Exhibit 5.1
VIX Movement (2006-2016)

Implied volatility

The simple average, EWMA and GARCH methods all try to forecast volatility using past volatility data. A somewhat different approach to estimate volatility is to start with the market price of a traded instrument and find out the corresponding value of the volatility. In a well functioning market, this estimate, called "implied volatility" is likely to be more useful than volatility based on historical data. In the Black Scholes Model, to be discussed later in the Chapter, to be discussed later in the chapter, all the option data including the price can be keyed in. Then by a process of trial and error, we can calculate implied volatility. The real utility of implied volatility lies in using the data available for already traded instruments to price a new instrument about to be launched. For such an instrument, past data, in any case would not exist. The implied volatility would be a good way of valuing the instrument.

Modelling stock price movements

Whenever we want to measure the market risk of a derivative, we have to understand the pattern of movement of the underlying. Thus we should be able to model the price of the underlying. Here, we look briefly at how modeling can be done when the underlying is a stock. Some of the most celebrated modeling work has been done for stocks but this work can be extended to other asset classes too.

Before we look at the modeling techniques, we need to gain a basic understanding of stochastic processes. When the value of a variable changes over time in an uncertain way, we say the variable follows a *stochastic* process. In a *discrete time* stochastic process, the value of the variable changes only at certain fixed points in time. In case of a *continuous time* stochastic variable, the changes can take place at any time. Stochastic processes may involve discrete or continuous variables. As their names suggest, discrete variables can only take discrete values while continuous variables can take any value. The continuous variable continuous time stochastic process comes in handy while describing stock price movements. Let us now describe some stochastic processes commonly used for stock price modeling.

Markov Process

In a *Markov process*, the past cannot be used to predict the future. Stock prices are usually assumed to follow a Markov process. This means all the past data have been discounted by the current stock price.

Let us elaborate this through a simple example provided by Paul Wilmott in his book "Quantitative Finance." Suppose we have a coin tossing game such that for every head we gain \$1 and for every tail, we lose \$1. Then the expected value of the gains after *i* tosses will be zero. For every toss, the expected value of the gain is zero. Suppose we use S_i to denote the total amount of money we have actually won upto and including the i^{th} toss. Then the expected value of S_i is zero. On the other hand, let us say we have already had 4 tosses and S_4 is the total amount of money we have actually won. The expected value of the fifth toss is zero. Thus the expected value after five tosses given the value after 4 tosses is nothing but S_4. When the expected value equals the current value of the variable, we call it a *martingale*, a special case of the Markov process. The distribution of the value of the random variable, S_i conditional upon all the past events only depends on the previous value, $S_{i-1.}$ This is the *Markov property*. The random walk has no memory beyond where it is now.

Wiener Process

A stochastic Markov process with mean change = 0 and variance = 1 per year is called a *Wiener process*. A variable *z* follows a Wiener process if the following conditions hold:

- The change Δz during a small period of time Δt is given by Δz = εΔt, where ε is a standard normal random variable with mean = 0 and std devn = 1.
- The values of Δz for any two different short intervals of time, Δt are independent.

- Mean of Δz = 0
- Variance of Δz = Δt or Standard deviation = √ Δt

Illustration

If a variable follows a Wiener process and has an initial value of 20, at the end of one year, the variable will be normally distributed with a mean of 20 and a standard deviation of 1.0. For a period of 5 years, the mean will remain 20 but the standard deviation will be √5. This is an extension of the principle we used earlier in the chapter to "scale" volatility as we go further out in time.

Generalized Wiener process.
Here the mean does not remain constant. Instead, it "drifts." This is unlike the basic Wiener process which has a drift rate of 0 and variance of 1. The generalized Wiener process can be written as:

$$dx = a\,dt + b\,dz$$

or

$$dx = a\,dt + b\varepsilon\sqrt{\Delta t}$$

Where "a" is the drift rate.

Illustration
Suppose the value of a variable is currently 40. The drift rate is 10 per year while the variance is 900 per year. At the end of 1 year the variable will be normally distributed, with a mean of 40+10 = 50 and a std deviation of $\sqrt{900}$= 30. At the end of 6 months, the variable will be normally distributed with a mean of 40+5 = 45 and std devn of $30\sqrt{.5}$ = 21.21.

Ito Process
An *Ito process* goes one step beyond the generalized Wiener process. The Ito process is nothing but a generalized Wiener process in which each of the parameters, a, b, is a function of both the underlying variable *x* and time, *t*. In an Ito process, the expected drift rate and variance rate of an Ito process are both liable to change over time.

$$\Delta x = a(x, t)\,\Delta t + b(x, t)\,\varepsilon\sqrt{\Delta t}$$

We will later discuss a process called Ito's Lemma that will come in handy while developing the Black Scholes equation. But let us for now get back to how stock prices can be modeled.

Brownian Motion
In the coin tossing experiment, we saw an interesting phenomenon. The expected winnings after any number of tosses is just the amount we already hold. As we mentioned earlier, this is called the *Martingale* property.

The *quadratic variation* of a random walk[5] is defined by
$[(S_1 – S_0)^2 + (S_2 – S_1)^2 + \ldots + (S_i – S_{i-1})^2]$

For each toss, the outcome is + \$1 or - \$1. So for the coin tossing experiment, each of the terms in the bracket will be $(1)^2$ or $(-1)^2$ i.e., exactly equal to 1. Since there are *i* terms within the square bracket, the quadratic variation is nothing but *i.*

Let us now advance the discussion by bringing in the time element. Suppose we have *n* tosses in the allowed time, *t*. We define the game in such a way that each time we toss

[5] Paul Wilmott on Quantitative Finance, www.wilmott.com

the coin, we may gain or lose an amount of $\sqrt{\frac{t}{n}}$. Now each term in the small bracket is

$\left[\sqrt{\frac{t}{n}}\right]^2$ or $\left[\sqrt{\frac{-t}{n}}\right]^2 = \frac{t}{n}$

Since there are *n* tosses, the quadratic variation is $(\frac{t}{n})$ (n) = t.

Thus the expected value of the pay off is zero and that of the variance is *t*.

The limiting process as time steps go to zero is called Brownian motion.

Geometric Brownian Motion

Brownian motion is useful but needs to be modified to make it useful for modelling stock prices. We need to introduce drift into Brownian motion. The most widely used model of stock price behaviour is given by the equation:

$$\frac{dS}{S} = \mu dt + \sigma dz$$

σ is the volatility of the stock price.
μ is the expected return.

This model is called *Geometric Brownian motion*. The first term on the right is the expected return which is somewhat predictable and the second is the stochastic component, which is somewhat unpredictable. In general, many variables can be broken down into a predictable deterministic component and a risky stochastic or random component. When we construct a risk free portfolio, our aim will be to eliminate the stochastic component. The component which moves linearly with time is deterministic and has no risk.

Illustration

Suppose a stock has a volatility of 20% per annum and provides an expected return of 15% per annum with continuous compounding. The process for the stock price can be written as:

$$\frac{dS}{S} = .15dt + .20dz$$

or $$\frac{\Delta S}{S} = .15\,\Delta t + .20\,\Delta z$$

or $$\frac{\Delta S}{S} = .15\,\Delta t + .20\,\varepsilon\sqrt{\Delta t}$$

If the time interval = 1 week = $\frac{1}{52}$ = .0192 years and the initial stock price is 50.

$$\Delta S = 50\,(.15 \times .0192 + .20\,\varepsilon\,\sqrt{.0192})$$
$$= .144 + 1.3856\,\varepsilon$$

To get a good intuitive understanding of Geometric Brownian motion, we draw on the work of Neil A Chriss. Readers are strongly advised to refer to his book, "Black Scholes and beyond" to get an intuitive common sense understanding of stock price modeling in general and the Black Scholes option pricing model in particular.

Consider a heavy particle suspended in a medium of light particles. These particles move around and crash into the heavy article. Each collision slightly displaces the heavy particle. The direction and magnitude of this displacement is random. It is independent of other collisions. Using statistical jargon, we can describe each collision as an *independent, identically distributed random event.*

The stock price is equivalent to the heavy article. Trades are equivalent to the light particles. We can expect stock prices will change in proportion to their size as the returns we expect do not change with the stock prices. Thus we would expect 20% return on Reliance shares whether they are trading at Rs. 50 or Rs. 500. So the expected price change will depend on the current price of the stock.

So we write:

$$\Delta s = S\,(\mu dt + \sigma dz).$$

Because we "scale" by S, it is called Geometric Brownian Motion.

In the short run, the return of the stock price is normally distributed. The mean of the distribution is $\mu\Delta t$. The std devn is $\sigma\sqrt{\Delta t}$. μ is the instantaneous expected return. σ is the instantaneous standard deviation.

In the long term, things are different. Let S be the stock price at time, t. Let μ be the instantaneous mean. Let σ be the instantaneous standard deviation.

The return on S between now (time t) and future time, T is normally distributed with a mean of $(\mu-\sigma^2/2)\,(T-t)$ and std devn of $\sigma\sqrt{T-t}$. Why do we write $(\mu-\sigma^2/2)$ and not μ? What is the intuitive explanation?

We need to first understand that volatility tends to depress the returns below what the short term returns suggest. Expected returns reduce because volatility jumps do not cancel themselves. A 5% jump multiplies the current stock price by 1.05. A 5% fall multiplies the amount by .95. If a 5% jump is followed by a 5% fall or vice versa, the stock price will reach 0.9975, not 1! In general, if a positive return x (x being defined in decimal terms) is followed by a negative return x, the price will reach (1+x) (1-x) = 1- x^2

How do we estimate the value of x? Consider a random variable x. We can calculate the variance of x as follows:

$\sigma^2 = E[x^2] - \{E[x]\}^2 = E[x^2]$ (assuming E[x] = 0, ie., ups and downs in *x* cancel out)

Thus the expected value of x^2 is the variance. But the amount by which the returns are depressed when a positive movement of x is followed by an equal negative movement is x^2. For two moves, the depression is x^2. So we could say that the average depression per move is $x^2/2$. But the expected value of x^2 is σ^2. $\left(\frac{1}{n}\sum x^2 = \sigma^2\right)$ So we can write $\sigma^2/2$ as the expected value of the amount by which the returns fall from the mean. That is why we write $(\mu-\sigma^2/2)$ and not μ.

[6]Can we make some prediction about the kind of distribution followed by the stock price under the assumption of a Geometric Brownian Motion? Let us begin with the assumption that the stock returns are normally distributed.

Let us first scale the returns to cover a period of one year.

Annualised return from t_0 to T = $\frac{1}{T-t_0} ln \frac{S_T}{S_{t_0}}$

S_T = future price S_{t0} = current price, $T-t_0$ is expressed in years.

Annualised return = $\frac{1}{T-t_0} lnS_T - \frac{1}{T-t_0} lnS_{t_0}$

Let us define random variable X = $\frac{1}{T-t_0} lnS_T - \frac{1}{T-t_0} lnS_{t_0}$

Let us define a new random variable

$$X + \frac{1}{T-t_0} lnS_{t_0}$$

The second term of the expression is a constant. So the basic characteristics of the distribution are not affected. Only the mean changes.

Also $X + \frac{1}{T-t_0} lnS_{t0} = \frac{1}{T-t_0} lnS_T$

or $(T-t_0) X + ln\, S_{t0} = ln\, S_T$

[6] Draws heavily from the work of Neil A Chriss, "Black Scholes and beyond – Option pricing models," McGraw Hill, 1997.

X is normally distributed. This means $ln\ S_T$ is normally distributed or S_T is lognormally distributed. So the price of a stock following a Geometric Brownian Motion is lognormally distributed.

The mean return on S from time t to time T is $(T-t)(r-\sigma^2/2)$, while the std devn is $\sigma\sqrt{T-t}$
The return on S from time t to T = $ln\ S_T/S_t$

So the random variable $\frac{ln\frac{S_T}{S_t}-(T-t)(r-\frac{\sigma^2}{2})}{\sigma\sqrt{T-t}}$ is normally distributed with mean = 0 and std devn = 1. In other words, we have converted a normal random variable into a standard normal random variable. This will make it easy for us to do computations using the standard normal tables.

Suppose a call option on the stock with strike price, K is in the money at expiration. That means the stock price exceeds the strike price. We want to estimate the probability of this happening. The required condition can be written as:

$$S_T \geq K$$

$$\Rightarrow \quad S_T/S_t \geq K/S_t$$

$$\Rightarrow \quad ln\,(S_T/S_t) \geq ln\,(K/S_t)$$

$$\Rightarrow \quad \frac{ln\frac{S_T}{S_t}-(T-t)(r-\frac{\sigma^2}{2})}{\sigma\sqrt{T-t}} \geq \frac{ln\frac{K}{S_T}-(T-t)(r-\frac{\sigma^2}{2})}{\sigma\sqrt{T-t}}$$

$$\Rightarrow \quad \frac{ln\frac{S_t}{S_T}+(T-t)(r-\frac{\sigma^2}{2})}{\sigma\sqrt{T-t}} \leq \frac{ln\frac{S_T}{K}+(T-t)(r-\frac{\sigma^2}{2})}{\sigma\sqrt{T-t}}$$

(Taking the negative of both sides and noting that $ln\frac{K}{S_T}=-ln\frac{S_T}{K}$ and $ln\frac{S_T}{S_t}=-ln\frac{S_t}{S_T}$

The probability of the stock price exceeding the strike price can be written as:

$$P\,(S_T \geq K) = N[\frac{(ln\frac{S_T}{K}+(T-t)(r-\frac{\sigma^2}{2})}{\sigma\sqrt{T-t}}]$$

$$\text{Or}\ P\,(S_T \geq K) = N\left[\frac{(ln\frac{S_T}{k})+(r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right]$$

This expression reminds us of the Black Scholes formula! The term within brackets is referred to as d_2 in the Black Scholes model. Indeed, GBM is central to Black Scholes pricing. GBM assumes stock returns are normally distributed. But empirical data reveals that large movements in stock price are more likely than a normally distributed stock price model suggests. The likelihood of returns near the mean and of large returns is greater than that predicted by GBM while other returns tend to be less likely. Research also indicates that monthly and quarterly volatilities are higher than annual volatility. Daily volatilities are lower than annual volatilities. So stock returns do not scale as they are supposed to.

Ito's lemma[7]

Let us move closer to the Black Scholes formula. Black and Scholes formulated a partial differential equation which they later solved, with the help of Merton by setting up boundary conditions. To understand the basis for their differential equation, we need to appreciate Ito's lemma. Consider G, a function of x. The change in G for a small change is x can be written as:

$$\Delta G = \frac{dG}{dx}\Delta x$$

We can understand this intuitively by stating that the change in *G* is nothing but the rate of change with respect to *x* multiplied by the change in *x*.

If we want a more precise estimate, we can use the Taylor series:

$$\Delta G = \frac{dG}{dx}\Delta x + \frac{1}{2}\frac{d^2G}{dx^2}(\Delta x)^2 + \frac{1}{6}\frac{d^3G}{dx^3}(\Delta x)^3 + ...$$

Now suppose G is a function of two variables, x and t. We will have to work with partial derivatives. This means we must differentiate with respect to one variable at a time, keeping the other variable constant. We could write:

$$\Delta G = \frac{\partial G}{dx}\Delta x + \frac{\partial G}{dt}\Delta t$$

Again, if we want to get a more accurate estimate, we could use the Taylor series:

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}(\Delta x)^2 + \frac{\partial^2 G}{\partial x dt}(\Delta x)(\Delta t) + \frac{1}{2}\frac{\partial^2 G}{\partial t^2}(\Delta t)^2 + ...$$

Note: $\left[\frac{1}{2}\frac{\partial^2 G}{\partial x \partial t}(\Delta x)(\Delta t) + \frac{1}{2}\frac{\partial^2 G}{\partial t \partial x}(\Delta x)(\Delta t) = \frac{\partial^2 G}{\partial x \partial t}(\Delta x)(\Delta t)\right]$

Suppose we have a variable x that follows the Ito process.

[7] This section and the following draw heavily from John C Hull, "Option, Futures and Other Derivatives," Prentice Hall, 2006.

$dx = a(x,t)\,dt + b(x,t)\,dz$

or $\Delta x = a(x,t)\,\Delta t + b(x,t)\varepsilon\sqrt{\Delta t}$

or $\Delta x = a\,\Delta t + b\,\varepsilon\sqrt{\Delta t}$

ε follows a standard normal distribution, with mean = 0 and standard deviation = 1.

We can write $(\Delta x)^2 = b^2\varepsilon^2\,\Delta t$ + other terms where the power of Δt is higher than 1.

If we ignore these terms (as Δt approaches zero) assuming they are too small, we can write:

$$\Delta x^2 = b^2\varepsilon^2\,\Delta t$$

All the other terms have Δt with greater power. They can be ignored. But Δx^2 itself is big enough and cannot be ignored.

Let us now go back to G and write:

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}(\Delta x)^2$$

But $\Delta x^2 = b^2\varepsilon^2\,\Delta t$ as we just saw a little earlier.

It can be shown (beyond the scope of this book) that the expected value of $\varepsilon^2\,\Delta t$ is Δt, as Δt becomes very small.

Thus $(\Delta x)^2 = b^2\Delta t$

Since we are approaching the limiting case, we replace ΔG by dG, Δx by dx and Δt by dt. So we can write the equation for change in G as:

$$dG = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial t}dt + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2 dt$$

But $dx = a(x,t)\,dt + b(x,t)\,dz$

So we can rewrite:

$$dG = \frac{\partial G}{\partial x}(adt + bdz) + \frac{\partial G}{\partial t}dt + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2 dt$$

$$= (a\frac{\partial G}{\partial x} + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2)dt + b\frac{\partial G}{\partial x}dz$$

This is called Ito's lemma.

The Black Scholes differential equation

The Ito's lemma is very useful when it comes to framing the Black Scholes differential equation.

Let us assume that the stock price follows Geometric Brownian motion, i.e.,

$$\frac{\Delta s}{s} = \mu \Delta t + \sigma \varepsilon \sqrt{\Delta t}$$

Or Δs = μsΔt + σsΔz

Let *f* be the price of a call option written on the stock whose price is modeled as S. *f* is a function of S and t. The change in stock price is given by:- ΔS = a (s,t) dt +b (s,t) ds.

Applying Ito's lemma, we can relate the change in *f* to the change in S .

Comparing with the general expression for Ito's Lemma, we get:

G = f, a = μs and b = σs, *x* = s,

or $$\Delta f = \left\{ \frac{\partial f}{\partial s}\mu s + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial s^2}\sigma^2 s^2 \right\} \Delta t + \frac{\partial f}{\partial s}\sigma s \Delta z$$

Our aim is to create a risk free portfolio whose value does not depend on S, the stochastic variable. Suppose we create a portfolio with a long position of $\frac{\partial f}{\partial s}$ shares and a short position of one call option. The value of the portfolio will be:-

$$\pi = -f + \frac{\partial f}{\partial s} s$$

Value refers to the net positive investment made. So a purchase gets a plus sign and a short sale gets a negative sign.

We will see later that $\frac{\partial f}{\partial s}$ is nothing but *delta* and the technique used to create a risk free portfolio is called *delta hedging*. Assume delta is constant over a short period of time.

Change in the value of the portfolio will be:

$$\Delta \pi = -\Delta f + \frac{\partial f}{\partial s}\Delta s$$

$$= -\left(\frac{\partial f}{\partial s}\mu s + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial s^2}\sigma^2 s^2 \right)\Delta t - \frac{\partial f}{\partial s}\sigma S \Delta z + \frac{\partial f}{\partial s}\Delta s$$

But Δs = μSΔt+σSΔz

or $\Delta\pi$ = $-\frac{\partial f}{\partial s}\mu s\Delta t-\frac{\partial f}{\partial t}\Delta t-\frac{1}{2}\frac{\partial^2 f}{\partial s^2}\sigma^2 s^2\Delta t-\frac{\partial f}{\partial s}\sigma s\Delta z+\frac{\partial f}{\partial s}(\mu s\Delta t+\sigma s\Delta z)$

= $-\frac{\partial f}{\partial s}\mu S\Delta t-\frac{\partial f}{\partial t}\Delta t-\frac{1}{2}\frac{\partial^2 f}{\partial s^2}\sigma^2 s^2\Delta t-\frac{\partial f}{\partial s}\sigma S\Delta z+\frac{\partial f}{\partial s}\mu s\Delta t+\frac{\partial f}{\partial s}\sigma s\Delta z$

or $\Delta\pi$ = $-\frac{\partial f}{\partial t}\Delta t-\frac{1}{2}\frac{\partial^2 f}{\partial s^2}\sigma^2 s^2\Delta t$

= $-\left(\frac{\partial f}{\partial t}+\frac{1}{2}\frac{\partial^2 f}{\partial s^2}\sigma^2 s^2\right)\Delta t$

This equation does not have a Δs term. It is a riskless portfolio, with the stochastic or risky component having been eliminated. The total return depends only on the time. That means the return on the portfolio is the same as that on other short term risk free securities. Otherwise, arbitrage would be possible. So we could write the change in value of the portfolio as:

$\Delta\pi = r\pi\,\Delta t$ where r is the risk free rate. (Because this is a risk free portfolio)

But $\pi = -f + \frac{\partial f}{\partial s}s$

or $\Delta\pi = r(-f+\frac{\partial f}{\partial s}s)\Delta t$

Also $\Delta\pi = -(\frac{\partial f}{\partial t}+\frac{1}{2}\frac{\partial^2 f}{\partial s^2}\sigma^2 s^2)\Delta t$

So $-\left(\frac{\partial f}{\partial t}+\frac{1}{2}\frac{\partial^2 f}{\partial s^2}\sigma^2 s^2\right)\Delta t = r\left(-f+\frac{\partial f}{\partial s}s\right)\Delta t$

or $rf = \frac{\partial f}{\partial t}+\frac{1}{2}\frac{\partial^2 f}{\partial s^2}\sigma^2 s^2 + r\frac{\partial f}{\partial s}s$

or $rf = \frac{\partial f}{\partial t}+rs\frac{\partial f}{\partial s}+\frac{1}{2}\sigma^2 s^2\frac{\partial^2 f}{\partial s^2}$

This is the Black Scholes differential equation.

It must be remembered that the portfolio used in deriving the Black Scholes differential equation is riskless only for a very short period of time when $\frac{\partial f}{\partial s}$ is constant. With change in stock price and passage of time, $\frac{\partial f}{\partial s}$ can change. So the portfolio will have to be continuously rebalanced to achieve what is called a perfectly hedged or zero delta position. This is also called *dynamic hedging*.

Risk neutral valuation

The variables in the Black Scholes differential equation are current stock price, time, stock price volatility and the risk free rate of interest. These variables are independent of the risk preferences of investors. Nowhere does the expected return, μ feature in the equation. This approach is called *risk neutral valuation*.

Risk neutral valuation is a very important concept in derivatives pricing theory. Essentially we adjust the probability of upward and downward movement of the price of the underlying asset in such a way that the value of the portfolio remains the same irrespective of the market outcome. If the outcome becomes independent of the market fluctuations, we can equate the return on the portfolio we have created with that of a risk free portfolio. The key benefit we enjoy with this approach is that the risk free rate is reasonably well known whereas the risk adjusted rate is subjective and contextual and hence difficult to establish.

The "probabilities" that we work out using the Black Scholes Model are not the actual probability of the stock being above or below the strike price of the option at expiration. The risk neutral probability is only a "notional probability" that we calculate by assuming that the underlying asset earns the risk free rate of return. As Delianedis and Geske[8] mention, risk neutral probability not only simplifies option valuation but also provides a deeper understanding of option models. To get the actual probability, we need to work out the expected return of the underlying asset. While that is not an easy task, the risk free rate of return can be more easily estimated. And as Duffie and Pan mention[9], the consistency of the risk neutral approach with efficient capital markets does not mean that investors are actually risk neutral. The actual risk represented by a position would typically differ from that represented by models based on risk neutral valuation. Over a short period, when markets are not very volatile and there is no jump risk, risk neutral behaviour and actual behaviour are quite similar. But the distinction between the two is significant over longer time horizons.

Illustration

Suppose there is an investment which yields 30% in a favorable scenario and -10% in an unfavorable scenario. The probability of a favorable scenario is 50% and that of an unfavorable one is also 50%. So the expected return is 10%. There is a risk free

[8] "Credit Risk and Risk Neutral Probabilities: Information about rating migrations and defaults," Working paper, May 1999.
[9] "An overview of value-at-risk," Working paper, January 21, 1997.

portfolio with a return of 5% that gives the same utility as the risky portfolio. What is the risk neutral probability?

Let p be the risk neutral probability of a favourable scenario happening.

Then	.05	=	p (.30) + (1 – p) (- .10)
or	.4p	=	.15
or	p	=	.375.

So the risk neutral probabilities are .375 for a favourable scenario and .625 for an unfavourable scenario.

Replicating portfolio

The Black and Scholes model is built on the principle of *portfolio replication*. Black and Scholes showed that a derivative security can be priced by replication i.e., by creating a portfolio of underlying assets whose returns exactly match the discounted expected value of the derivative's pay off at maturity using risk neutral probability. This portfolio may consist of say an option and the underlying stock. The Black Scholes model tells us how to create a perfect hedge. The cost of setting up the hedged position and any additional costs incurred in maintaining it can be calculated ahead of time. The kind of hedging associated with Black Scholes involves only set up costs, no maintenance costs. Such an approach is called a self financing strategy.

Let us introduce one more term now, *complete market*. A market is said to be complete if any derivative can be replicated using only the underlying assets. In a complete market, any derivative, how much ever complex, has an unique arbitrage free price. It can be shown that a model of security prices is complete if and only if the model admits exactly one risk neutral probability. We shall explore this theme in more detail little later in the Chapter.

Computing risk neutral probability[10]

The Black Scholes model can be considered as nothing but the binomial model, in the limiting case where the time period becomes infinitesimal. A brief note on the binomial model is in order here. In the binomial model, the stock price can take only two values at the end of a time period. Consider a risk free bond and a derivative whose value X will either be X_u or X_d after a period of time. Let X = ΔS + a, i.e., X consists of Δ units of the underlying stock and '*a*' units of bond. At the end of a given period, the stock price can be u S (upper bound) or d S (lower bound), where d < r < u. The corresponding values of X are X_u and X_d respectively. The bond value will be *ar* at the end of the period, irrespective of the price of the security.

[10] Draws heavily from Rangajaran K Sundaram, "Equivalent Martingale measures and risk neutral pricing: An expository note," Journal of Derivatives, Fall 1997.

Let $\Delta = \frac{X_u - X_d}{uS - dS}$

and $a = \frac{uX_d - dX_u}{r(u-d)}$

In a bullish scenario, the value of the portfolio will be:

$\Delta uS + ar$ $= \frac{X_u - X_d}{uS - dS} uS + \frac{uX_d - dX_u}{r(u-d)} r$

$= \frac{X_u - X_d}{u - d} u + \frac{uX_d - dX_u}{u-d}$

$= \frac{uX_u - uX_d + uX_d - dX_u}{u-d}$

$= \frac{X_u(u-d)}{(u-d)} = X_u$

Similarly, in a bearish scenario, the value of the portfolio will be:

$\Delta dS + ar$ $= \frac{X_u - X_d}{uS - dS} dS + \frac{uX_d - dX_u}{r(u-d)} r$

$= \frac{dX_u - dX_d + uX_d - dX_u}{u-d}$

$= \frac{X_d(u-d)}{(u-d)} = X_d$

Thus the portfolio we have constructed, is a replicating portfolio. It replicates the pay off of the derivative. The portfolio itself consists of the underlying stock and a risk free bond. Now, it is easy to find out the current value of the derivative.

The current value of the portfolio is:

$X = \Delta S + a$ (by definition)

$= \frac{X_u - X_d}{uS - dS} S + \frac{uX_d - dX_u}{r(u-d)}$

$= \frac{(X_u - X_d)r + uX_d - dX_u}{r(u-d)}$

$$= \frac{1}{r}\left[\frac{rX_u - rX_d + uX_d - dX_u}{(u-d)}\right]$$

$$= \frac{1}{r}\left[\frac{r-d}{u-d}X_u + \frac{u-r}{u-d}X_d\right]$$

So the risk neutral probabilities are $\frac{r-d}{u-d}$ (for the stock price reaching u S) and $\frac{u-r}{u-d}$ (for the stock price reaching d S).

We can derive the risk neutral probability in a different way. If the expected returns for the portfolio are to be the same as that for the risk free bond

	pu + (1-p) d	=	r
or	pu + d – pd	=	r
or	pu – pd	=	r-d
or	p = $\frac{r-d}{u-d}$		
and	1 – p	=	$\frac{u-r}{u-d}$

Pricing derivatives using risk neutral probabilities is computationally convenient. The risk neutral probability depends only on the underlying assets in the problem and not on the particular claim being valued. A risk neutral probability must satisfy two conditions:

- The prices that occur under the risk neutral probability must be identical to those that occur in the original model.
- The expected returns on all assets in the model should be the same.

In some models, the risk neutral probability is uniquely defined but in general, a model may allow more than one risk neutral probability. Or in some cases, it may not admit one at all. The risk neutral probability does not exist if and only if there are arbitrage opportunities. Multiple risk neutral probabilities can occur if and only if there are contingent claims in the model that cannot be priced by arbitrage because they are not replicable.

More about risk neutral probability

What is the economic interpretation of risk neutral probability? To answer this question, let us first define an *Arrow security*. An Arrow security is a security associated with a particular future state of the world that pays $1 if that state occurs and zero otherwise. All other contingent claims and derivative securities can be expressed in terms of portfolios of Arrow securities and priced accordingly.

As Rangarajan Sundaram[11] explains, the state price associated with a particular state is simply the risk neutral probability of the state discounted at the risk free rate. Risk neutral probabilities are just the prices of certain contingent claims. Multiplying a claim's pay offs in a particular state by the state price is exactly the same as multiplying it by the discounted risk neutral probability of that state. Thus there is a one-to-one relationship between risk neutral probabilities and state prices. *Identifying risk neutral probabilities, essentially means identifying the state prices.*

We have already seen that in the binomial model with two possibilities, i.e., the stock price changing to uS and dS, the risk neutral probability is given by:

$$p = \frac{r-d}{u-d}$$

and $$1-p = \frac{u-r}{u-d}$$

The discounted risk neutral probabilities are given by:

$$\frac{p}{r} = \frac{1}{r}\frac{(r-d)}{(u-d)}$$

and $$\frac{1-p}{r} = \frac{1}{r}\frac{(u-r)}{(u-d)}$$

Let us get back to Arrow securities. The pay offs in the case of Arrow security are 0 and 1. In the Binomial model

$$X = \frac{1}{r}\left[\frac{r-d}{u-d}X_u + \frac{u-r}{u-d}X_d\right]$$

Now, the statement made by Sundaram is clear.
A model does not permit arbitrage if and only if it admits at least one risk neutral probability. There is no arbitrage only if $u > r > d$.

If we put $X_u = 1$; $X_d = 0$, we get one state price

$$X_u = \frac{1}{r}\frac{r-d}{u-d}$$

If we put $X_u = 0$; $X_d = 1$, we get another state price

$$X_d = \frac{1}{r}\frac{u-r}{u-d}$$

By comparing with the earlier equation, we get:

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[11] Journal of Derivatives, Fall 1997.

$$p_u = \frac{p}{r}; \qquad p_d = \frac{1-p}{r}$$

Thus, the binomial model is consistent with risk neutral valuation.
Let us now take the example of a model that does not admit any risk neutral probability. Consider a portfolio with two risky assets and a bond. Assume that after one period, the prices can be either : rB, ($u_1 S_1$, $d_1 S_1$) or rB, ($u_2 S_2$, $d_2 S_2$). If we want this to be a risk free portfolio, we can write:

$$p\, u_1 + (1-p)\, d_1 = r$$
$$p\, u_2 + (1-p)\, d_2 = r$$

Simplifying these equations, we get:

$p\,(u_1-d_1) = r - d_1$ and $p\,(u_2-d_2) = r - d_2$

or $p = \dfrac{r-d_1}{u_1-d_1} = \dfrac{r-d_2}{u_2-d_2}$

We can choose r, u_1, d_1, u_2, d_2 such that this last equation is violated. So no risk neutral probability exists.

Let us now examine a model where there can be more than one risk neutral probability. Suppose S can take three values at the end of the period, uS, mS and dS, where u > m > d. To be a risk free portfolio, we can write:

$$p_u u + p_d d + p_m m = r$$
with $p_u + p_d + p_m = 1$

There are two equations and three unknowns. So there are infinitely many solutions here.
Thus the trinomial model is also not complete. Let us illustrate this.

Suppose we have invested in “a” shares of stock and “b” dollars of bond.

We can then write:

$$a\,u\,S + b\,r = X_u$$
$$a\,m\,S + b\,r = X_m$$
$$a\,d\,S + b\,r = X_d$$

If we combine the first two equations, we get:

$$a\,u\,S - a\,m\,S = X_u - X_m$$

or $a = \dfrac{X_u - X_m}{uS - mS}$

Similarly from the second and third equations, we get:

$$a = \frac{X_m - X_d}{mS - dS}$$

Suppose we put $X_u = X_m = 1$ and $X_d = 0$. Then the two equations are inconsistent. In one case, a = 0 and in the other case, $a = \frac{1}{mS - dS}$

Let us make a final point before we close the discussion on risk neutral probability. In the binomial model, we had two states uS and dS at the end of a period. The risk free rate was r.

Without making any explicit attempt to arrive at the probability of state uS and probability of state dS, we get:

$$X = \frac{1}{r}\left[\frac{r-d}{u-d}X_u + \frac{u-r}{u-d}X_d\right]$$

So that $p = \frac{r-d}{u-d}$

and $1 - p = \frac{u-r}{u-d}$

Why did we not have to explicitly assume the probabilities for state uS and dS and compute them? We have made it a risk free portfolio. The composition of the portfolio is the same irrespective of whether one state occurs 50% of the time or 70% the time. In short, calculating the price of the replicating portfolio does not require knowledge of the probabilities of the two states. It certainly looks puzzling! The dilemma is resolved if we intuitively understand that *the probabilities of the two states are embedded into the current stock price*. As the current stock price changes, the probabilities of the two states will also change.

The Black Scholes Merton Formula[12]

We have already seen the Black Scholes differential equation. Now let us move on to the Black Scholes Merton formula. Suppose the stock price follows the Geometric Brownian motion $dS = \mu Sdt + \sigma sdz$.

We saw a little earlier that the price of a call option, f could be related to the stock price S through the equation:

[12] Draws from John C Hull, "Options, Futures and Other Derivatives," Prentice Hall, 2006.

$$\Delta f = \left(\frac{\partial f}{\partial s}\mu s + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial s^2}\sigma^2 S^2\right)\Delta t + \frac{\partial f}{\partial s}\sigma s \Delta z$$

If *f = lnS,* $\frac{\partial f}{\partial s} = \frac{1}{s}$, $\frac{\partial^2 f}{\partial s^2} = \frac{-1}{s^2}$, $\frac{\partial f}{\partial t} = 0$

or $$\Delta f = \frac{1}{s}\mu s \Delta t - \frac{1}{2s^2}\sigma^2 s^2 \Delta t + \frac{1}{s}\sigma s \Delta z$$

or $$\Delta f = \left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma \Delta z$$

or $$df = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dz$$

What this shows is that f = *l*ns follows a generalized Wiener process with a constant drift rate of $\left(\mu - \frac{\sigma^2}{2}\right)$ and a constant variance rate of σ^2. We have already seen earlier why the drift is $\left(\mu - \frac{\sigma^2}{2}\right)$ and not μ.

The change in S between 0 and future time T is normally distributed with mean $\left(\mu - \frac{\sigma^2}{2}\right)$T and variance $\sigma^2 T$. or standard deviation $\sigma\sqrt{T}$.

$$ln\, S_T - lnS_0 \sim \emptyset \left[\left(\mu - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right]$$

or $$ln\, S_T \sim \emptyset \left[lnS_0 + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right]$$

Now we are ready to write the Black-Scholes-Merton formulae for valuing European call and put options. Let *c* denote the price of the European call option and *p* that of the European put option.

European call option: c = $S\,N(d_1) - e^{-rT}(K)\,N(d_2)$

European put option: p = $e^{-rT}(K)\,N(-d_2) - S\,N(-d_1)$

Where:

S	=	current value of stock price
N (d)	=	cumulative normal distribution at d
K	=	strike price

r = continuously compounded risk free rate of interest
T = time to expiration
σ = volatility
d_1, d_2 are more complex terms as mentioned below:

$$d_1 = \frac{ln\frac{S}{K} + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{ln\frac{S}{K} + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

$$= d_1 - \sigma\sqrt{T}$$

Let us take a simple illustration. Let S = 100, K = 110, r = 9.50%
T = 6 months = .5 years, σ = 20%

$$\text{Then } d_1 = \frac{ln\frac{100}{110} + (.0950 + \frac{.2^2}{2})(.5)}{.2\sqrt{.5}}$$

$$= \frac{-.09531 + .0575}{.1414}$$

$$= -.2674$$

$$d_2 = -.2674 - (.2)\sqrt{.5}$$

$$= -.4088$$

From tables,

$N(d_1)$ = .3950
$N(d_2)$ = .3417
C = $SN(d_1) - Ke^{-rt} N(d_2)$
= $(100)(.3950) - (110)\, e^{-(.0950)(.5)}\, (.3417)$
= 39.50 – 35.83
= 3.67

Let us sum up the assumptions of the original Black Scholes Model.

- The model applies only to European options.
- No dividends are paid.
- The market is complete, efficient and frictionless.
- Options and securities are perfectly divisible.
- There are no riskless arbitrage opportunities.
- Borrowing and lending can take place at the risk free rate.
- The risk free rate is constant during the life of the option.

- Stock prices follow a lognormal distribution.

Thus the original Black Scholes model operates within a restrictive framework. Fortunately, the model can be extended by relaxing many of these assumptions. A lot of work has indeed been done in this regard.

Illustration

A European call option has the following characteristics. Current price of stock: 50; strike price: 45; risk free rate of return: 5%; maturity: 1 year; annual volatility: 25%. What is the value of the call?

$$d_1 = \frac{ln(\frac{50}{45}) + [.05 + (.5)(.25)^2](1)}{.25\sqrt{1}}$$

or d_1 = (.1054 + .08125) / .25 = .7466

d_2 = .7464 - (.25) $\sqrt{1}$ = .4966

$N(d_1)$ = .7723

$N(d_2)$ = .6902

C = (50) (.7723) – ($45e^{-.05}$ x .6902)

= 38.615 – 29.544

≈ 9.071

Illustration

A stock trades for 60. The annual std devn is 10%. The continuously compounded risk free rate is 5%. Calculate the value of both call and put if the exercise price is 60 and maturity is 1 year.

- $$d_1 = \frac{ln(\frac{60}{60}) + [.05 + (.5) \times (.10)^2](1)}{.10\sqrt{1}}$$
- = 0.55

$d_2 = d_1 - \sigma\sqrt{T}$ = .55 - 10√1 = .45

$N(d_1)$ = .7088 $N(d_2)$ =.6736

Value of call = (S_0) $(N(d_1))$ – [Ke^{-rt} × $N(d_2)$]

= (60) (.7088) – [$60e^{-(.05)}$ × .6736]

$= 42.53 - 38.44 \quad = \quad 4.09$

We can calculate the value of the put by applying the put call parity.

According to put call parity, $p + s = c + ke^{-rT}$

where p is the value of the put option
c is the value of the call option
s is the current stock price
k is the strike price for both the options
r is the risk free rate
T is the time to maturity

Value of put = $4.09 - 60 + 60\, e^{-(.05)(1)} = 1.16$

Conclusion

In this chapter we have attempted to understand the sources of market risk. We have also covered briefly some of the tools and techniques available to measure market risk. A basic understanding of how stock prices can be modeled has also been provided. We have examined one of the key building blocks of market risk measurement, risk neutral valuation. The coverage has been kept basic and as non technical as possible. Readers must consult more specialized books to get a more indepth understanding of the topics covered in this chapter.

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